Mini Review

BIOMEDICAL Journal of Scientific & Technical Research

ISSN: 2574 -1241

DOI: 10.26717/BJSTR.2022.47.007522

Oscillatory Behavior of Second Order Nonlinear Difference Equations with Mixed Neutral Terms

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ARTICLE INFO

Received: November 28, 2022

Published: December 09, 2022

Citation: Said R Grace. Oscillatory Behavior of Second Order Nonlinear Difference Equations with Mixed Neutral Terms. Biomed J Sci & Tech Res 47(4)-2022. BJSTR. MS.ID.007522.

ABSTRACT

This paper deals with the oscillation of second order nonlinear difference equations with mixed nonlinear neutral terms. The purpose of the present paper is the linearization of the considered equation in the sense that we would deduce oscillation of studied equation from that of the linear form and to provide new oscillation criteria via comparison with first order equations whose oscillatory behavior are known. The obtained results are new, improve and correlate many of the known oscillation criteria appeared in the literature. The results are illustrated by some examples.

Keywords: Second Order; Nonlinear; Neutral; Mixed Type; Oscillation AMS (MOS) Classification 34N05, 39A10

Introduction

This paper is concerned with oscillatory behavior of all solutions of the nonlinear second order difference equations with mixed neutral terms of the form

$$\Delta\left(a(n)(\Delta y(n))^{\alpha}\right)+q(n)w^{\alpha}\,(n-m+1)+p(n)w^{\alpha}\,(n+m^{*})=0 \qquad (1.1)$$

where

$$\Delta w(t) = w(t+1) - w(t) \ and \ y(n)=w(n)+p_1\,(n)w^{\beta}\,(n-k)-p_2\,(n)w^{\delta}\,(n-k).$$

We shall assume that

(I) α,β,γ,μ" and " δ are the ratios of positive odd integers, α≥1.

(II) {p_1 (n)},{p_2 (n)}, {q (n)} and {p (n)} are sequences of positive real numbers.

(III) k, m, m*are positive real numbers with h (n) = n – m + k + 1 and $h^{*}(n)=n+m^{*}+k.$

We let

$$A(v,u)=\sum_{s=u}^{v-1}\frac{1}{a\frac{1}{\alpha}(s)} and A(n,n_1)=\sum_{s=n_1}^{n-1}\frac{1}{a\frac{1}{\alpha}(s)}\to\infty \qquad (1.2)$$

for $n \geq n \geq n_0 \ \theta=\max\{k,m-1,m^{*}+1\}$. By a solution of equation (1.1), we mean a real sequence {x(t)} defined for all t≥t_0-θ and satisfies equation (1.1) for all t≥t_0. A solution of equation (1.1) is called oscillatory if its terms are neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. If all solutions of the equation are oscillatory then the equation itself called oscillatory. In recent years, there has been a great interest in establishing criteria for the oscillation and asymptotic behavior of solutions of various classes of second-order difference equations, see [1-15] and the references cited therein. However, to the best of our knowledge, there are no results for second-order difference equations with mixed neutral terms of type (1.1). More exactly, existing literature does not provide any criteria which ensure oscillation of all solutions of equations (1.1). The aim of the

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present paper is the linearization of equation (1.1) in the sense that we would deduce oscillation of studied equation from that of the linear form and to provide new oscillation criteria (taking the linear form of equation (1.1) into account) via comparison with first order equations whose oscillatory behavior are known. The obtained results are new, improve and correlate many of the known oscillation criteria appeared in the literature for equation (1.1).

Main Results

In this section we study some oscillation criteria for equation (1.1) when β<1" and " δ>1. We start with the following fundamental result. See [10, Lemma 1], and for the proof of (I), see [15, Lemma 2.2].

Lemma 2.1. Let {q (n)} be a sequence of positive real numbers, m and m*are positive real number and f: R→ R is a continuous non-decreasing function, and x f(x) > 0 for x≠0,

(I) The first order delay differential inequality

$\vartriangle y(n)+q(n)f(y(n-m+1))\le 0$ has an eventually positive solution, so does the delay equation

$$\vartriangle y(n)+q(t)f(y(t-m+1))\le 0$$

(II) The first order advanced differential inequality

$\vartriangle y(t)-q(n)f(y(n+m^{*}))\ge 0$ has an eventually positive solution, so does the delay equation $\vartriangle y(n)-q(n)f(y(n+m^{*}))=0$ Lemma 2.2. [13]. If X and Y are nonnegative, then

$$X^{\lambda}+(\lambda-1)Y^{\lambda}-\lambda XY^{\lambda-1}\ge 0\, for\, \lambda>1, \quad (2.1)$$

$$X^{\lambda}-(1-\lambda)Y^{\lambda}-\lambda XY^{\lambda-1}\le 0\, for 0<\lambda<1 \quad (2.2)$$

where equality is held if and only if X = Y.

In what follows, we let

$$g_2(n):=(\delta-1)\frac{\delta}{\delta_1-\delta p_2}\frac{1}{1-\delta(n)},g_1(n):=(1-\beta)\frac{\beta}{\beta^{1-\beta}}b\frac{\beta}{\beta-1(n)}p_1^{\frac{1}{1-\beta}}(n)$$

$$p(n)=\frac{p(n)}{(p_2(h^{*}(n))^{\frac{\alpha}{\delta}}} and Q(n)=\frac{q(n)}{(p_2(h(n))^{\frac{\alpha}{\delta}}},n\ge n_1$$

"for some n"≥n_0," where " {b (n)} is a sequence of positive real numbers.

Now, we present the following oscillation result. Theorem 2.1. Let β<1" and " δ>1, conditions (i) – (iv) and (1.3) hold. Assume that there exist positive sequences (b (t)} and positive real numbers $k_1$and k_2 such that k_1 < m-k_1 and k_2 < m* + k +1 such that

$$\lim_{n\to\infty}[g_1(n)+g_2(n)]=0 \quad (2.3)$$

If the first order advanced equation

$$\vartriangle Z(n)-\frac{1}{\alpha}p(n)A^{\frac{\alpha}{\delta}}(h^{*}(n),p(n))z^{1-\alpha+\frac{\alpha}{\delta}}(p(n))=0 \quad (2.4)$$

is oscillatory, where ρ(n) = n +m*+k-$k_{2>}$t, and assume that there exists a number θ∈(0,1) such that both the delay equations

$$\vartriangle w(n)+\theta A^{\alpha}(n-m+1,t_1)q(n)W(n-m+1)=0 \quad (2.5)$$

for some n_1≥n_0 and

$$\vartriangle x(n)+\frac{1}{\alpha}Q(n)A^{\frac{\alpha}{\delta}}(\{n),h(n))X^{1-\alpha+\frac{\alpha}{\delta}}(\{(n))=0 \quad (2.6)$$

where, ξ(n)=n-m+k_1+1<t

Proof. Let {w (n)} be a nonoscillatory solution of equation (1.1), say w (n) > 0, w(n-k)>0 ,w(n-m+1)>0 , and w(n+ m*+1)>0 for n≥n_1 for some n_1≥n. It follows from equation (1.1) that

$$\vartriangle((a(n)(\vartriangle y(n)^{\alpha})=-q(n)x^{\alpha}(n-m+1))-p(n)x^{\alpha}(n+m)\le 0 \quad (2.7)$$

Hence $((a(n)(\vartriangle y(n)^{\alpha}$ is nonincreasing and of one sign. That is, there exists a $n_2\ge n_1$ such that $\vartriangle y(n)>0$ or $\vartriangle y(n)<0$ for $n\ge n2$ Now, we see that $\vartriangle(a(n)(\vartriangle y(n)^{\alpha})=\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))^{\alpha}$ Taking the difference of the above inequality, we get

$$\vartriangle(a(n)(\vartriangle y(n)^{\alpha})\ge\frac{1}{\alpha}(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))^{\alpha-1})\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n)) \quad (2.8)$$

From equation (1.1) one can easily see that

$$\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))+\frac{1}{\alpha}(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))^{\alpha-1})q(n)x^{\alpha}(n-m+1)+p(n)x^{\alpha}(n+m^{*})\le 0 \quad (2.9)$$

From (2.9) one can easily get

$$\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))+\frac{1}{\alpha}(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))^{1-\alpha})q(n)x^{\alpha}(n-m+1)\le 0 \quad (2.10)$$

We shall distinguish the following four cases:

(I) $y(n)>0 and \vartriangle y(n)<0$

(II) $y(n)>0 and \vartriangle y(n)>0$

(III) $y(n)<0 and \vartriangle y(n)>0$

(IV) $y(n) < 0 and \vartriangle y(n) < 0$

First, we consider Case (I): Since $\vartriangle y(n) < 0$ for $n \geq n_2$ By condition (1.2), we conclude that $\lim_{n\to\infty} y(n) = -\infty$ a contradiction to the fact that y (n) is eventually positive. Next, we consider Case (II). Now, from the definition of y (n), we get

$$y(n) = w(n) + (b(n)w(n-k) - p_2(n)w^{\delta}(n-k) + (p_1(n)w^{\delta}(n-k) - b(n)w(n-k))$$

Or

$$w(n) = y(n) + (b(n)w(n-k) - p_2(n)w^{\delta}(n-k) - (p_1(n)w^{\delta}(n-k) - b(n)w(n-k))$$

If we apply (2.1) with

$$\lambda = \delta > 1, X = p_2^{\frac{1}{\delta}}(n)w(n) and Y = (\frac{1}{\delta}b(n)p_2^{\frac{-1}{\delta}}(n))\frac{1}{\delta - 1}$$

we have

$$(p_1(n)w^{\beta}(n-k) - b(n)w(n-k)) \leq (1-\beta)\frac{\beta}{\beta^{1-\beta}}b\frac{\beta}{\beta-1}(n)p_1^{\frac{1}{1-\beta}}(n) := g_1(n)$$

Thus, we see that

$$w(n) \geq [1 - \frac{g_1(n)+g_2(n)}{y(n)}y(n)$$

Since y (t) in nondecreasing, there exists a constant C > 0 such that y(n)≥C , and so, we have

$$w(n) \geq [1 - \frac{g_1(n)+g_2(n)}{c}y(n)$$

Now, there exists a constant c_1∈(0,1) such that

$$w(n) \geq c_1 y(n) \quad (2.11)$$

Using (2.11) in (2.10), we have

$$\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n)) + \frac{c_1^{\alpha}}{\alpha}(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n))^{1-\alpha})q(n)y^{\alpha}(n-m+1) \leq 0 \quad (2.12)$$

Clearly, we see that

$$y(n) \geq \sum_{s-n_1}^{n-1}\alpha^{-\frac{1}{\alpha}}(s)(\alpha^{\frac{1}{\alpha}}(s)\vartriangle y(s)) \geq A(n,n_1)(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n)).$$

Using this inequality in (2.12), we find

$$\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n)) + \frac{c_1^{\alpha}}{\alpha}(\frac{y(n)}{A(n,n_1)}^{1-\alpha}q(n)y^{\alpha}(n-m+1) \leq 0 \quad (2.13)$$

It is easy to see that the function $\frac{y(n)}{A(n,n_1)}$ is a nonincreasing and so, we get

$$\vartriangle(\alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n)) + \frac{c_1^{\alpha}}{\alpha}A^{1-\alpha}(n-m+1,n_1)q(n)y(n-m+1) \leq 0$$

(2.14) We let $v(n) = \alpha^{\frac{1}{\alpha}}(n)\vartriangle y(n)$ and so, we see that $\vartriangle y(n) = \frac{v(n)}{\alpha^{\frac{1}{\alpha}}(n)}$ and $y(n) \geq A(n,n_1)v(n)$ Using this inequality in (2.14), we have

$$\vartriangle v(n) + \frac{c_1^{\alpha}}{\alpha}A^{\alpha}(n-m+1,t_1)q(n)v(n-m+1) \leq 0$$

It follows from Lemma 2.1. (I) that the corresponding differential equation (2.5) also has a positive solution, a contradiction. Next, we consider the cases when y (n) < 0 fort≥t_2.

Let

$$z(n) = -y(n) = -w(n) - p_1(n)w^{\beta}(n-k) + p_2(n)w^{\delta}(n-k) \leq p_2(n)w^{\delta}(n-k)$$

Or

$$w(t-k) \geq [1 - (\frac{z(t)}{p_2(t)})^{\frac{1}{\delta}} \quad \text{or} \quad x(t) \geq [1 - (\frac{z(t+k)}{p_2(t+k)})^{\frac{1}{\delta}}$$

and so, Now, we consider Case (III). Clearly, we see that $\vartriangle z(n) = -\Delta y(n) < 0$ for $n \geq n_1$

$$\vartriangle((a(n)(\vartriangle z(n)^{\alpha}) = q(n)w^{\alpha}(n-m+1) + c(n)w^{\alpha}(n+m^*)$$

$$\geq \frac{q(n)}{p_2^{\frac{\alpha}{\delta}}(h(n))}z^{\frac{\alpha}{\delta}}(h(n)) + \frac{p(n)}{p_2^{\frac{\alpha}{\delta}}(h^*(n))}z^{\frac{\alpha}{\delta}}(h^*(n)) \quad (2.15)$$

As in the proof of Theorem 2.1, we see that

$$\vartriangle((a(n)(\vartriangle z(n))^{\alpha} = \vartriangle a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha} \geq \alpha(a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha-1})\vartriangle a^{\frac{1}{\alpha}}(n)\vartriangle z(n)).$$

From equation (1.1) one can easily see that $\vartriangle a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha} \geq \frac{1}{\alpha}(a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha-1})(q(n)x^{\alpha}(n-m+1) + p(n)x^{\alpha}(n+m^*)$ or

$$\vartriangle a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha} \geq \frac{1}{\alpha}(a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha-1})\left(Q(n)z^{\frac{\alpha}{\delta}}(h(n)) + p(n)x^{\alpha}(n+m^*)\right)\vartriangle a^{\frac{1}{\alpha}}(n)\vartriangle z(n)). \quad (2.16)$$

From (2.16) one can easily find that

$$\vartriangle a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha} \geq \frac{1}{\alpha}(a^{\frac{1}{\alpha}}(n)\vartriangle z(n))^{\alpha-1})Q(n)z^{\frac{\alpha}{\delta}}(h(n)) \quad (2.17)$$

We consider Case (III) where z(n)>0 and Δz(n)<0 Now, for n_1≤u≤v, we may write Δz(n)=-Δy(n)<0 for n≥n_1

$$z(u) - z(v) = -\sum_{s-u}^{v-1}a^{\frac{-1}{\alpha}}(s)(a(s)(\vartriangle z(s))^{\alpha})^{\frac{1}{\alpha}} \geq A(v,u)\left(-a^{\frac{1}{\alpha}}(v)(\vartriangle z(v))\right)$$

We let u = h (n) and v = ξ(n) in the above inequality we see that

$$z(h(n)) \geq A(\xi(n),h(n))(-a^{\frac{1}{\alpha}}(\xi(n)))(\Delta z(\xi(n)) \quad (2.19)$$

Using (2.19) in (2.17), we have

$$\Delta(a^{\frac{1}{\alpha}}(n)\Delta z(n))^{\alpha} + \frac{1}{\alpha}(a^{\frac{1}{\alpha}}(n)\Delta z(n))^{\alpha-1})(Q(n)A(\xi(n),h(n))(-a^{\frac{1}{\alpha}}(\xi(n)))(\Delta z(\xi(n)))^{\frac{\alpha}{\delta}} \leq 0$$

which finally takes the form

$$\Delta w(n) + \frac{1}{\alpha}(Q(n)A^{\frac{\alpha}{\delta}}(\xi(n),h(n))W^{\frac{\alpha}{\delta}-\alpha+1}(\xi(n)) \leq 0$$

where W (n) = $a^{\frac{1}{\alpha}}(n)\Delta z(n)$. The rest of the proof is similar to that of Case (I) and hence is omitted. Next, we consider Case (IV), i.e., z(n)>0 and Δz(n)>0. Now,

$$z(n) \geq z(n) - z(n-k_2) = \sum_{s-n-k_2}^{n-1} a^{\frac{-1}{\alpha}}(s)a^{\frac{1}{\alpha}}(s)\Delta z(n)$$

$$\geq A(n,n-k_2)a^{\frac{1}{\alpha}}(n-k_2)\Delta z(n-k_2).$$

Or

$$\geq z(h^*(n)) \geq A(h^*(n),p(n)a^{\frac{1}{\alpha}}p(n)\Delta z(p(tn))$$

Using this inequality in the inequality (2.18) we get

$$\Delta(a^{\frac{1}{\alpha}}(n)\Delta z(n)) \geq \frac{1}{\alpha}(a^{\frac{1}{\alpha}}(n)\Delta z(n))^{\alpha-1})p(n)\left(A(h^*(n),p(n))a^{\frac{1}{\alpha}}p(n)\Delta zp(n))\right)^{\frac{\alpha}{\delta}}.$$

It is easy to find that

$$\Delta(a^{\frac{1}{\alpha}}(n)\Delta z(n)) \geq \frac{1}{\alpha}p(n)A^{\frac{\alpha}{\delta}}(h^*(n)),p(n)Z^{1-\alpha+\frac{\alpha}{\delta}}(p(n)).$$

It follows from Lemma 2.1. (II) that the corresponding differential equation (2.4) also has a positive solution. This contradiction completes the proof.

Remark 2.1.

We note that the results of this paper can be extended easily to the more general equations of the for

$$\Delta(a(n)\Delta(w(n)+p_1(n)w^{\beta}(n-k)-p_2(n)w^{\delta}(n-k))^{\alpha})+q(n)w^{\gamma}(n-m+1)+p(n)w^{\mu}(n+m^*)=0$$

where the coefficients are the same as in equation (1,1) with γ and μ are the ratio of positive odd integers. The details are left to the reader. For the special case when α= δ, i.e. ,the equation

$$\Delta(a(n)\Delta(w(n)+p_1(n)w^{\beta}(n-k)-p_2(n)w^{\alpha}(n-k))^{\alpha})+q(n)w^{\alpha}(n+m-1)+p(n)w^{\alpha}(n+m^*)=0$$

(2.20) and when α=1,i.e.,the equation

$$\Delta\left(a(n)\Delta(w(n)+p_1(n)w^{\beta}(n-k)-p_2(n)w(n-k)\right)+q(n)w(n+m-1)+p(n)x(n+m^*)=0$$

(2.21) we have the following interesting results Corollary 2.1. Let the hypotheses of Theorem 2.1 hold with equations (2.4) and (2.6) are replaced (respectively) by:

$$\Delta z(n) - \frac{1}{\alpha}p(n)A(h^*(n),p(t)Z^{z-\alpha}p(n)) = 0 \quad (2.22)$$

$$\Delta w(n) + \frac{1}{\alpha}Q(n)A(\xi(n),h(n)w^{z-\alpha}(\xi(n)) = 0 \quad (2.23)$$

Then equation (2.20) is oscillatory.

We also have the following result from corollary 2.1 for equation (2.21): Corollary 2.2. Let the hypotheses of Theorem 2.1 hold with equations (2.4) and (2.6) are replaced (respectively) by

$$\Delta z(n) - p(n)A(h^*(n),p(n)Z(p(n)) = 0 \quad (2.24)$$

$$\Delta w(n) + Q(n)A(\xi(n),h(n)W(\xi(n)) = 0 \quad (2.25)$$

Then equation (2.21) is oscillatory.

The following corollary is to employ some integral conditions rather than the oscillatory behavior of first order equations involved. Corollary 2.3. Let the hypotheses of Theorem 2.1 hold. If

$$\liminf_{n\to\infty}\sum_{s=n}^{p(n)-1} p(s)A(h^*(s),p(s))\Big\{\begin{smallmatrix}=\infty\\>a\end{smallmatrix}\ \overset{where 1<\alpha<2}{\frac{(p(n)-1)^{p(n)}}{p^{p(n)}(n)}}\ when \alpha=1 \quad (2.26)$$

$$\liminf_{n\to\infty}\sum_{s=n-m+1}^{n-1} q(s)A^{\alpha}(s-m+1,n_1) = \infty when 1 \leq \alpha \quad (2.27)$$

And

$$\liminf_{n\to\infty}\sum_{s=\xi(n)}^{n-1} Q(s)A(\xi(s),h(s))\Big\{\begin{smallmatrix}=\infty\\>a\end{smallmatrix}\ \overset{where 1<\alpha<2}{\frac{(\xi^{\xi(n)}(n)}{(\xi(n)+1^{\xi(n)+1}}}\ when \alpha=1 \quad (2.28)$$

then equation (2.20) is oscillatory. The following example is illustrative: Example 2.1. Consider the mixed neutral second order differential equations

$$\Delta(n^2\Delta(w(n)+\frac{1}{n}w^{\frac{1}{3}}(n-k)-w^{\frac{5}{3}}(n-k))\frac{5}{3})+q(n)w^{\frac{5}{3}}(n+m-1)+p(n)w^{\frac{5}{3}}(n+m^*)=0 \quad (2.29)$$

And

$$\Delta(n^3\Delta\left((w(n)+\frac{1}{n}w^{\frac{1}{3}}(n-k)-w(n-k)\right)+q(n)w(n-m+1)+p(n)w(n+m^*)=0 \quad (2.30)$$

Here we have $\{q(n)\}$ and $\{p(n)\}$ are are positive sequences of real numbers ,a (n)= n^3,A(n,n_1)=

$$\sum_{s=n_1}^{n-1}\frac{1}{s}$$

p_1 (n)=1/n→0 as n→∞ and p_2 (n)=1=b(n), α =5/3= δ and β=1/3, k, m,m*are positive real numbers with h (n) = n – m + k + 1 and h* (n)=n+m*+k and positive real numbers k_1 and k_2 such that k_1 < m-k_1 and k_2 < m* + k +1 with ρ(n) = n +m*+k-k_2>n and ξ(n)=n-m+k_1+1<n. It is easy to see for appropriate function p and

q and the numbers k_1 and k_2 that all conditions of Corollary 2.3 are satisfied and hence every solution x (t) of equation (2.29) (respectively (2.30)) is oscillatory.

Remarks

The paper is presented in a form which is essentially and of high degree of generality. It will be of interest to study these results for the higher order of the

$$\Delta((a(n)(\vartriangle n^{*-1} y(n))^{\alpha}) + q(n)w^{\alpha}(n-m+1) + c(n)w^{\alpha}(m+m^{*}) = 0 \quad (2.31)$$

and

$$\Delta((a(n)(\vartriangle n^{*-1} y(n))^{\alpha}) = q(n)w^{\alpha}(n-m+1) + c(n)w^{\alpha}(m+m^{*}). \quad (2.32)$$

Competing Interests

The author declare that they have no competing interests.

Data Availability Statement

Not Applicable.

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ISSN: 2574-1241

DOI: 10.26717/BJSTR.2022.47.007522

Said R Grace. Biomed J Sci & Tech Res